Equivalent equation analysis of a kinetic relaxation model

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Introduction

We consider the scalar conservation law in 2 dimensions

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = 0, \qquad (\mathcal{E})$$

where

- $w(\boldsymbol{x}, t) \in \mathbb{R}$,
- $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$,
- $\boldsymbol{q}(w) = \begin{pmatrix} q_1(w(\boldsymbol{x}, t)) \\ q_2(w(\boldsymbol{x}, t)) \end{pmatrix}$.

Introduction

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = 0. \qquad (\mathcal{E})$$

- We only have one variable $w \in \mathbb{R}$. We use a kinetic scheme $D2Qn_v$ which approximate $(\mathcal{E})$ with n_v equations with variables denoted $\boldsymbol{f} \in \mathbb{R}^{n_v}$.
- Kinetic models are efficient numerical scheme which use transport at **constant velocities**. However, it can be difficult to analyze them.
- The solution of this equation given by a kinetic model can be approximate by an **equivalent equation**, which will have n_v variables.
- The analysis of this equivalent equation gives us information on the **stability** and the **boundary conditions**.

Plan

Kinetic approximation

$$\partial_t w + \nabla \cdot \boldsymbol{q}(w) = 0 \qquad (\mathcal{E})$$

We consider the BGK kinetic model

$$\partial_t f_i + \nabla \cdot (\boldsymbol{\lambda}_i f_i) = \frac{1}{\epsilon}\left(f_i^{eq} - f_i\right), \qquad \text{for } i = 1, \dots, n_v, \qquad (\mathcal{K})$$

where

- $\boldsymbol{\lambda}_i$ are the kinetic velocities,
- $\boldsymbol{f} = (f_i)$ is the kinetic unknown,
- $\boldsymbol{f}^{eq} = (f_i^{eq})$ is the equilibrium kinetic vector which satisfy the **consistency relations**

$$w = \sum_{i=1}^{n_v} f_i^{eq} \quad \text{and} \quad \boldsymbol{q}(w) = \sum_{i=1}^{n_v} \boldsymbol{\lambda_i} f_i^{eq}.$$

In the limit $\epsilon \to 0$, $\sum_{i=1}^{n_v} f_i$ tends to the solution w.

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = \frac{1}{\epsilon}(f_i^{eq} - f_i)$$

When $\epsilon \to 0$, we have $f_i \to f_i^{eq}$.
By summing the n_v equations, we obtain

$$\sum_{i=1}^{n_v} \partial_t f_i + \sum_{i=1}^{n_v} \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = \frac{1}{\epsilon}\left(\sum_{i=1}^{n_v} f_i^{eq} - \sum_{i=1}^{n_v} f_i\right).$$

We took the limit when $\epsilon \to 0$, we have

$$\partial_t \left(\sum_{i=1}^{n_v} f_i^{eq}\right) + \boldsymbol{\nabla} \cdot \left(\sum_{i=1}^{n_v} \boldsymbol{\lambda}_i f_i^{eq}\right) = 0.$$

Using the consistency conditions, we finally obtain

$$\partial_t w + \nabla \cdot (\boldsymbol{q}(w)) = 0.$$

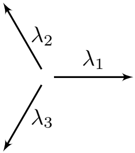
The kinetic velocities

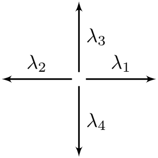
- In the $D2Q3$ model, we have $n_v = 3$ kinetic velocities

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} -\frac{\lambda}{2} \\ \frac{\lambda\sqrt{3}}{2} \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} -\frac{\lambda}{2} \\ -\frac{\lambda\sqrt{3}}{2} \end{pmatrix}.$$

- In the $D2Q4$ model, we have $n_v = 4$ velocities along the cartesian axes

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} -\lambda \\ 0 \end{pmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{pmatrix} 0 \\ \lambda \end{pmatrix}, \quad \boldsymbol{\lambda}_4 = \begin{pmatrix} 0 \\ -\lambda \end{pmatrix}.$$





The moments

The consistency conditions gives us the system

$$\begin{pmatrix} w \\ q_1(w) \\ q_2(w) \\ z_3^{eq} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda_{1,1} & \lambda_{2,1} & \lambda_{3,1} & \lambda_{4,1} \\ \lambda_{1,2} & \lambda_{2,2} & \lambda_{3,2} & \lambda_{4,2} \\ m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \end{pmatrix}}_{M} \begin{pmatrix} f_1^{eq} \\ f_2^{eq} \\ f_3^{eq} \\ f_4^{eq} \end{pmatrix}.$$

With the $D2Q4$ model, we are free to choose the third moment and its equilibrium. We choose:

$$m_{i,3} = (\lambda_{i,1})^2 - (\lambda_{i,2})^2 \qquad \text{and} \qquad z_3^{eq} = 0.$$

We note $\boldsymbol{g} = \begin{pmatrix} w \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}$ the variables such as

$$\boldsymbol{g} = M\boldsymbol{f} \tag{1}$$

Splitting method

To solve in time the kinetic model

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = \frac{1}{\epsilon}(f_i^{eq} - f_i), \qquad (\mathcal{K})$$

we apply a splitting method:

- **Transport step**:

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = 0. \qquad (\mathcal{T})$$

- **Relaxation step**:

$$\partial_t f_i = \frac{1}{\epsilon}(f_i^{eq} - f_i). \qquad (\mathcal{R})$$

We consider a time step $\Delta t > 0$. At each iteration, we solve $(\mathcal{T})$ and $(\mathcal{R})$ on Δt.

Transport step

The exact solutions of the n_v transport equations

$$\partial_t f_i + \boldsymbol{\lambda}_i \cdot \boldsymbol{\nabla} f_i = 0, \tag{$\mathcal{T}$}$$

write

$$f_i^*(\boldsymbol{x}, t + \Delta t) = f_i(\boldsymbol{x} - \Delta t \boldsymbol{\lambda_i}, t).$$

The resolution of $(\mathcal{T})$ for one time step can be written

$$\boldsymbol{f}^*(\boldsymbol{x}, t + \Delta t) = D(\Delta t)\boldsymbol{f}(\boldsymbol{x}, t),$$

with the translation operator

$$\left(\tau_i(h)v\right)(\boldsymbol{x}) = v(\boldsymbol{x} - h\boldsymbol{\lambda}_i),$$

and D the diagonal matrix operator $D(\Delta t) = \begin{pmatrix} \tau_1(\Delta t) & & \\ & \ddots & \\ & & \tau_{n_v}(\Delta t) \end{pmatrix}$.

Flux error

As we have $\boldsymbol{g} = M\boldsymbol{f}$, we can rewrite the transport step as

$$\boldsymbol{g}^*(\boldsymbol{x}, t + \Delta t) = MD(\Delta t)M^{-1}\boldsymbol{g}(\boldsymbol{x}, t).$$

We define the **flux error** as

$$y_k = z_k - q_k(w), \qquad \text{for } k = 1, 2.$$

The transport step in the moments $\boldsymbol{g} = (w, \boldsymbol{z})$ can be rewritten on the error flux $\boldsymbol{h} = (w, \boldsymbol{y})$

$$\boldsymbol{h}^*(\boldsymbol{x}, t + \Delta t) = \begin{pmatrix} w^*(\boldsymbol{x}, t + \Delta t) \\ y_1^*(\boldsymbol{x}, t + \Delta t) \\ y_2^*(\boldsymbol{x}, t + \Delta t) \\ z_3^*(\boldsymbol{x}, t + \Delta t) \end{pmatrix}$$

$$= MD(\Delta t)M^{-1} \begin{pmatrix} w(\boldsymbol{x}, t) \\ y_1(\boldsymbol{x}, t) + q_1(w(\boldsymbol{x}, t)) \\ y_2(\boldsymbol{x}, t) + q_2(w(\boldsymbol{x}, t)) \\ z_3(\boldsymbol{x}, t) \end{pmatrix} - \begin{pmatrix} 0 \\ q_1(w^*(\boldsymbol{x}, t + \Delta t)) \\ q_2(w^*(\boldsymbol{x}, t + \Delta t)) \\ 0 \end{pmatrix}$$

Relaxation step

We want to solve

$$\partial_t f_i = \frac{1}{\epsilon}(f_i^{eq} - f_i). \qquad (\mathcal{R})$$

We note

- f_i^n: the kinetic fields at time $t_n = n\Delta t$.
- f_i^*: the kinetic fields after the free transport step.
- $f_i^{*,eq}$: the equilibrium fields after the free transport step.

We approximate $(\mathcal{R})$ by the relaxation formula

$$f_i^{n+1} = f_i^* + \omega\left(f_i^{*,eq} - f_i^*\right), \quad \text{with } \omega \in [1,2].$$

By choosing $\omega = 2$ (justification of this choice below), we have

$$\begin{pmatrix} w(\boldsymbol{x}, t+\Delta t) \\ z_1(\boldsymbol{x}, t+\Delta t) \\ z_2(\boldsymbol{x}, t+\Delta t) \\ z_3(\boldsymbol{x}, t+\Delta t) \end{pmatrix} = \begin{pmatrix} w^*(\boldsymbol{x}, t+\Delta t) \\ 2q_1(w^*(\boldsymbol{x}, t+\Delta t)) - z_1^*(\boldsymbol{x}, t+\Delta t) \\ 2q_2(w^*(\boldsymbol{x}, t+\Delta t)) - z_2^*(\boldsymbol{x}, t+\Delta t) \\ -z_3^*(\boldsymbol{x}, t+\Delta t) \end{pmatrix}.$$

Plan

Equivalent equation

- When $\omega = 2$ and up to second order terms in Δt the equivalent equation of the $D2Q3$ scheme is:

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} q_1'(w) & 0 & 0 \\ 0 & \frac{\lambda}{2} - q_1'(w) & 0 \\ 0 & -q_2'(w) & -\frac{\lambda}{2} \end{pmatrix}}_{A_1} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix}$$
$$+ \underbrace{\begin{pmatrix} q_2'(w) & 0 & 0 \\ 0 & 0 & -\frac{\lambda}{2} - q_1'(w) \\ 0 & -\frac{\lambda}{2} & -q_2'(w) \end{pmatrix}}_{A_2} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \end{pmatrix} = 0.$$

- When $\omega = 1$ the equivalent equation is only a first order approximation.

▶ In green, we retrieve the initial equation $(\mathcal{E})$.

Numerical validation of the equivalent equation

We can compare

- y_1^{vf} and y_2^{vf} obtained by solving the equivalent equation (with a finite volume method, for instance),
- y_1^{kin} and y_2^{kin} obtained by $\boldsymbol{y}^{kin} = \sum_{i=1}^{3} \boldsymbol{\lambda_i} f_i - q(w)$ after solving the equation $(\mathcal{E})$ with the $D2Q3$ model.

We choose the parameters

- $\Omega = [0,1] \times [0,1]$,
- $q_1'(w) = 1$ and $q_2'(w) = 1$,
- $\lambda = 3$,
- a Gaussian initialization

$$w(\boldsymbol{x},0) = \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{x}_0^w\|^2}{2\sigma^2}\right) \quad \text{and} \quad y_k(\boldsymbol{x},0) = \exp\left(-\frac{\|\boldsymbol{x}-\boldsymbol{x}_0^y\|^2}{2\sigma^2}\right)$$

with $\sigma = 0.05$, $\boldsymbol{x}_0^w = (0.25, 0.25)$ and $\boldsymbol{x}_0^y = (0.5, 0.5)$.

Validation of the equivalent equation

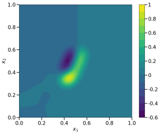
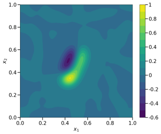
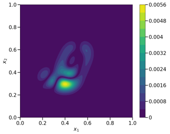
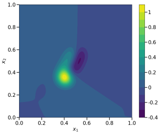
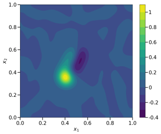
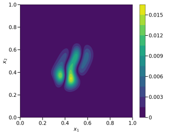
	$\boldsymbol{y}^{kin}$	$\boldsymbol{y}^{vf}$	$\|\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\|$
y_1			
y_2			

Table: Error fluxes $\boldsymbol{y}^{kin}$ and $\boldsymbol{y}^{vf}$ and the L^2 error $\|\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\|$ at $T_f = 0.06$ for a mesh of size 800×800.

$$\|\boldsymbol{y}_1^{kin} - \boldsymbol{y}_1^{vf}\| = 5.64567 \times 10^{-4} \quad \text{and} \quad \|\boldsymbol{y}_2^{kin} - \boldsymbol{y}_2^{vf}\| = 1.95625 \times 10^{-3}$$

- The equivalent equation is a good approximation of the scheme and therefore it gives useful information in its behavior.

Subcharacteristic stability condition

A classical result is the following subcharacteristic stability condition. If we consider $\omega \neq 2$ and a linear flux $q(w) = \begin{pmatrix} aw \\ bw \end{pmatrix}$, the equivalent equation is

$$\partial_t w + \nabla \cdot q(w) = \Delta t \left(\frac{1}{\omega} - \frac{1}{2} \right) \nabla \cdot (\mathcal{D} \nabla w) + O(\Delta t^2),$$

with the **diffusion matrix**

$$\mathcal{D} = \begin{pmatrix} \frac{\lambda}{2}(\lambda + a) - a^2 & -\frac{\lambda}{2} b - ab \\ -\frac{\lambda}{2} b - ab & \frac{\lambda}{2}(\lambda - a) - b^2 \end{pmatrix}.$$

The model is stable if this diffusion matrix is positive, that is if the eigenvalues of $\mathcal{D}$ are positive.

The **subcharacteristic stability condition** is

$$\frac{1}{2} \left(\lambda^2 - a^2 - b^2 \pm \sqrt{(a^2 + b^2)^2 + \lambda(-2a^3 + 6ab^2) + \lambda^2(a^2 + b^2)} \right) > 0.$$

Hyperbolicity condition

Proposition

If the subcharacteristic condition is satisfied then, the change of variable $\boldsymbol{h} = P\boldsymbol{m}$ symmetrizes the equivalent equation, which is thus a hyperbolic system with an entropy. We have

$$\partial_t P\boldsymbol{m} + A_1 P \partial_{x_1}\boldsymbol{m} + A_2 P \partial_{x_2}\boldsymbol{m} = 0,$$

with A_1 and A_2 the matrices of the equivalent equation and

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\lambda}{2}(\lambda + a) - a^2 & -\frac{\lambda}{2}b - ab \\ 0 & -\frac{\lambda}{2}b - ab & \frac{\lambda}{2}(\lambda - a) - b^2 \end{pmatrix}$$

is **hyperbolic**.

Equivalent equation

By the same Taylor expansion,

we get **the equivalent equation of the $D2Q4$ model**

$$\partial_t \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} + \underbrace{\begin{pmatrix} q_1'(w) & 0 & 0 & 0 \\ 0 & -q_1'(w) & 0 & \frac{1}{2} \\ 0 & -q_2'(w) & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 \end{pmatrix}}_{A_1} \partial_{x_1} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix}$$

$$+ \underbrace{\begin{pmatrix} q_2'(w) & 0 & 0 & 0 \\ 0 & 0 & -q_1'(w) & 0 \\ 0 & 0 & -q_2'(w) & -\frac{1}{2} \\ 0 & 0 & -\lambda^2 & 0 \end{pmatrix}}_{A_2} \partial_{x_2} \begin{pmatrix} w \\ y_1 \\ y_2 \\ z_3 \end{pmatrix} = 0.$$

Numerical validation of the equivalent equation

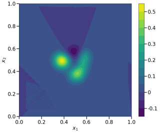
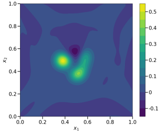
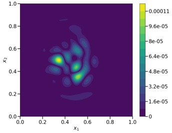
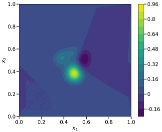
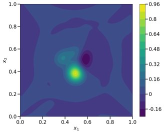
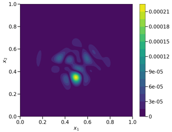
	$\boldsymbol{y}^{kin}$	$\boldsymbol{y}^{vf}$	$\|\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\|$
y_1			
y_2			

Table: Error fluxes $\boldsymbol{y}^{kin}$ and $\boldsymbol{y}^{vf}$ and the L^2 error $\|\boldsymbol{y}^{kin} - \boldsymbol{y}^{vf}\|$ at $T_f = 0.06$ for a mesh of size 800×800.

$$\|\boldsymbol{y}_1^{kin} - \boldsymbol{y}_1^{vf}\| = 1.21999 \times 10^{-5} \quad \text{and} \quad \|\boldsymbol{y}_2^{kin} - \boldsymbol{y}_2^{vf}\| = 1.57384 \times 10^{-5}$$

- The equivalent equation is a good approximation of the scheme and therefore it gives useful information in its behavior.

Subcharacteristic stability condition

If we consider a **linear flux** $q(w) = \begin{pmatrix} aw \\ bw \end{pmatrix}$, we have

$$\partial_t w + \nabla \cdot q(w) = \Delta t \left(\frac{1}{\omega} - \frac{1}{2} \right) \nabla \cdot (\mathcal{D} \nabla w) + O(\Delta t^2),$$

with the **diffusion matrix**

$$\mathcal{D} = \begin{pmatrix} \frac{\lambda^2}{2} - a^2 & -ab \\ -ab & \frac{\lambda^2}{2} - b^2 \end{pmatrix}.$$

The model is stable if this diffusion matrix is positive, that is if the eigenvalues of $\mathcal{D}$ are positive.

The **subcharacteristic condition for viscous stability** is

$$a^2 + b^2 \leq \frac{\lambda^2}{2}.$$

Hyperbolicity condition

Proposition

If $\lambda^2 > 4\max(a^2, b^2)$, then the system

$$\partial_t P\boldsymbol{m} + A_1 P\partial_{x_1}\boldsymbol{m} + A_2 P\partial_{x_2}\boldsymbol{m} = 0,$$

with A_1 and A_2 the matrices of the equivalent equation and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda^2}{2} - a^2 & -ab & \lambda^2 a \\ 0 & -ab & \frac{\lambda^2}{2} - b^2 & -\lambda^2 b \\ 0 & \lambda^2 a & -\lambda^2 b & \lambda^4 \end{pmatrix},$$

is **hyperbolic**.

This hyperbolicity condition is more restrictive than the viscous stability condition.

Plan

Boundary conditions

- In theory, the over-relaxation gives us a **second order accuracy**. We want to find adapted boundary condition, which gives us this accuracy.

- A first choice is to only impose boundary condition on w. But if we solve the equation with a kinetic model $D2Qn_v$, then we need (in general) more conditions. Therefore, we need **additional conditions on the variables y_1, y_2 (and z_3 for the $D2Q4$ model)**.

- Moreover, we can only impose w at the **inflow boundary**.

- In one dimension, the second order is achieved with a Dirichlet condition on w at the inflow border, and a Neumann condition on y at the outflow border (see [Drui *et al.*, 2019]).

Signs of the eigenvalues

We have the equivalent equation $\partial_t \boldsymbol{h} + A_1 \partial_{x_1} \boldsymbol{h} + A_2 \partial_{x_2} \boldsymbol{h} = 0$.
Let's note $n = (n_1, n_2)$ a normal vector.

A strategy is to impose the components in the basis of the eigenvectors of the matrix $A_1 n_1 + A_2 n_2$ when the associated eigenvalues are negative.

We choose

- a square geometry rotated of an angle $\frac{\pi}{10}$
- the initialization

$$w(x_1, x_2, t=0) = \begin{cases} 0 & \text{if } r(x_1, x_2) > 1, \\ (1 - r(x_1, x_2)^2)^5 & \text{otherwise.} \end{cases}$$

  with $r(x_1, x_2) = \frac{\sqrt{(x_1 - c_1)^2 + (x_2 - c_2)^2}}{\sigma}$ and $\sigma = 0.4$.
- $\lambda = 1,\ T = 1$, and $Nt = 25, 50, 100, 200$.

Signs of the eigenvalues

We consider 2 different test-cases :

1. The peak starts outside the geometry and arrives in the middle of the left border.
2. The peak starts in the middle of the square and arrives in the middle of the left border.

	1	2
a	$-0.5\cos(\pi/10+\pi)$	$-0.5\cos(\pi/10)$
b	$-0.5\sin(\pi/10+\pi)$	$-0.5\sin(\pi/10)$
c_1	$0.5+\cos(\pi/10+\pi)$	0.5
c_2	$0.5+\sin(\pi/10+\pi)$	0.5

Signs of the eigenvalues

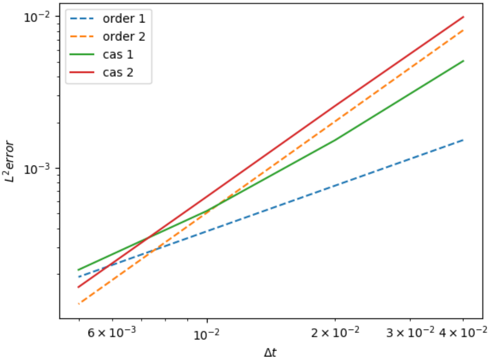


Figure: Error L^2 for the two test-case with the boundary conditions defined with the signs of the eigenvalues

We can observe that this boundary condition strategy does not give us a second-order accuracy for the first test-case, but it is at least **stable**.

Plan

Guiding-center model

Now, we consider the **guiding-center model** in 2 dimensions, which describes the drift of the plasma

$$\begin{cases} \partial_t \rho + \boldsymbol{v} \cdot \nabla \rho = 0, \\ -\Delta \phi = \rho, \end{cases}$$

where

- ρ is the **ion density**,
- ϕ is the **potential**,
- E is the **electric field** defined as $\boldsymbol{v}(\boldsymbol{x}, t) = (-\nabla \phi(\boldsymbol{x}, t))^{\perp} = E(\boldsymbol{x}, t)^{\perp}$.

We use a finite element solver on the same poloidal mesh to solve Poisson equation in the poloidal plane.

Initialization

We initialize the density with the continuous function

$$\rho(r, \theta, 0) = e^{-\frac{(r-r0)^2}{2\sigma^2}}(1 + \epsilon \cos(k\theta)),$$

with

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

We choose a ring geometry:

$$\Omega = \{(r\cos(\theta), r\sin(\theta)) \mid r_{min} \leq r \leq r_{max}, \\ 0 \leq \theta \leq 2\pi\},$$

with homogeneous Dirichlet boundary conditions on the potential ϕ at $r = r_{min}$ and $r = r_{max}$.

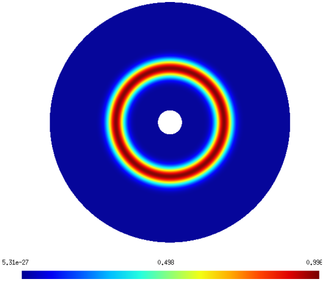


Figure: Initialization of the density at time $t = 0$, with $r_{min} = 1$, $r_{max} = 10$, $r_0 = 4.5$, $\sigma = 0.5$, $\epsilon = 10^{-6}$.

Instability rate

We can estimate the growth rate of the instability created by the perturbation, and compare it with the results of our solver:

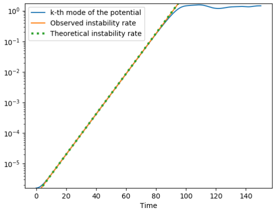


Figure: Instability rate observed compared to the theoretical one.

- Theoretical instability rate: 0.15215
- Observed instability rate: 0.15186

- The slope of the k^{th} Fourier mode of the potential is fitting the theoretical instability rate, which validates our computing.

2D Diocotron testcase

We choose $\Delta t = 0.0125$, $n_t = 8000$, $\lambda_p = 7$, $\omega = 1.999$, and $k = 2$.
We consider a mesh of size: 100×60.

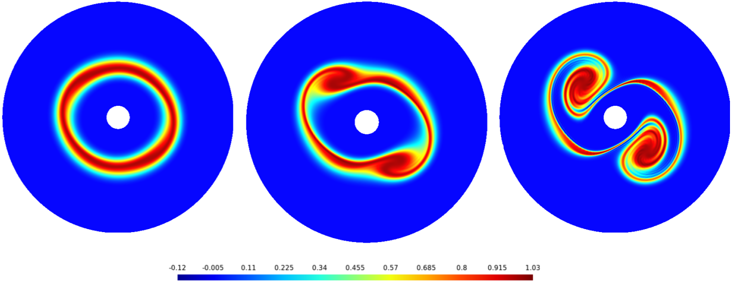


Figure: Densities obtained at time t = 80, t = 90 and t = 100.

3D model

We consider the model in 3 dimensions which describes the drift of the plasma inside a tokamak.

$$\begin{cases} \partial_t \rho + \nabla \cdot ((E \times e_z + B)\rho) = 0, \\ -\Delta_{x,y}\phi = \rho, \\ E = -\nabla_{x,y}\phi, \end{cases}$$

with

- ρ is the **density**,
- E is the **electric field**,
- $B = (-sin(\theta)e_x + cos(\theta)e_y)B_\theta + B_z e_z$ with θ the angle of the polar coordinates in the plane (x, y): a divergence free **magnetic field** (this is satisfied if B_θ and B_z are constants).

The $D3Q6$ model

- In the (x, y) planes, we do a $D2Q4$ model:

$$\boldsymbol{\lambda}_0 = \begin{pmatrix} \lambda_p \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\lambda}_1 = \begin{pmatrix} -\lambda_p \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{\lambda}_2 = \begin{pmatrix} 0 \\ \lambda_p \\ 0 \end{pmatrix}, \ \boldsymbol{\lambda}_3 = \begin{pmatrix} 0 \\ -\lambda_p \\ 0 \end{pmatrix}.$$

- In the z direction, we do a $D1Q2$ model:

$$\boldsymbol{\lambda}_4 = \begin{pmatrix} 0 \\ 0 \\ \lambda_z \end{pmatrix}, \quad \boldsymbol{\lambda}_5 = \begin{pmatrix} 0 \\ 0 \\ -\lambda_z \end{pmatrix}.$$

- We use an **unstructured mesh** in the (x, y) direction and a **periodic structured mesh** in the z direction.
- We solve the transport kinetic equations with a **Discontinuous Galerkin method** in (x, y) and a **characteristic method** in the z direction.
- The implementation is parallelized with OpenMP in the (x, y) planes and with MPI in the z direction.
- The solver is CFL-less.

Initialization of the Diocotron testcase

We initialize the density with

$$\rho(r,\theta,z,0) = e^{-\frac{(r-r_0)^2}{2\sigma^2}} \left(1 + \epsilon \cos\left(k\theta + lz\frac{2\pi}{L}\right)\right).$$

The computational domain is the cylinder

$$\Omega = \{(r\cos(\theta), r\sin(\theta), z) \mid r_{min} \le r \le r_{max}, 0 \le \theta \le 2\pi,\ 0 \le z \le L = 1\}.$$

We consider

- homogeneous Dirichlet boundary conditions on the potential ϕ,
- periodic boundary conditions on z.

3D Diocotron testcase

We choose $\Delta t = 0.0026$, $n_t = 38400$, $\omega = 1.99$, $n_p = 128$, $\lambda_p = 7$, $\lambda_z = 3$, $B_\theta = 0.1$, $B_z = 1$, $k = 2$ and $l = 1$.
In the poloidal plane, we took a mesh of size 80×50.

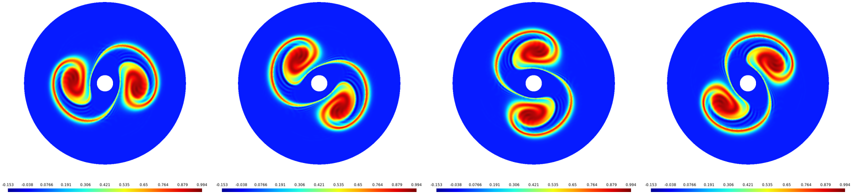


Figure: Density at $t_{max} = 100$ for the poloidal planes $z = 0$, $z = \frac{L}{4}$, $z = \frac{L}{2}$ and $z = \frac{3L}{4}$.

- Same density, but with a rotation.

Animation of the Diocotron testcase

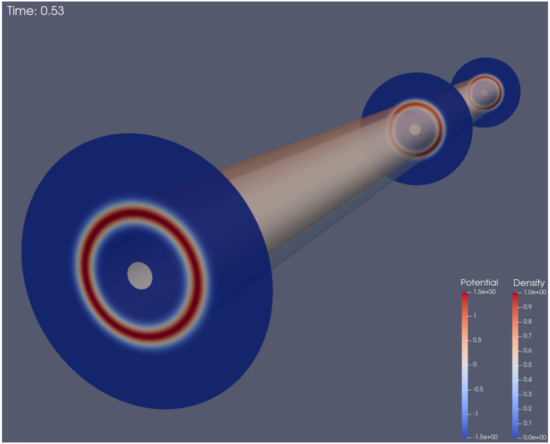


Figure: Evolution of the density in three poloidal planes.

Conclusion

- The equivalent equation of the kinetic models gives useful information about **stability** and **boundary conditions**.
- The kinetic scheme is **CFL-less** and can be used on **unstructured grids**.
- We have computed and tested it in an efficient way on a parallel computer.

Thank you for your attention !

References

Drui, Florence, Franck, Emmanuel, Helluy, Philippe, & Navoret, Laurent. 2019. An analysis of over-relaxation in a kinetic approximation of systems of conservation laws. *Comptes Rendus Mécanique*, **347**(3), 259–269.

Hyperbolicity condition

According to the equivalent equations, we have

$$\partial_t \boldsymbol{g} + A_1 \partial_{x_1} \boldsymbol{g} + A_2 \partial_{x_2} \boldsymbol{g} = 0,$$

with $A_1 = \begin{pmatrix} \frac{\lambda}{2} - a & 0 \\ -b & -\frac{\lambda}{2} \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -\frac{\lambda}{2} - a \\ -\frac{\lambda}{2} & -b \end{pmatrix}$.

Theorem

If $A_1 P$ and $A_2 P$ are symmetric and P is symmetric positive-definite, then

$$\partial_t P\boldsymbol{g} + A_1 P \partial_{x_1} \boldsymbol{g} + A_2 P \partial_{x_2} \boldsymbol{g} = 0$$

is hyperbolic.

Hyperbolicity condition $D2Q3$ (demonstration)

Demonstration :

We note $P = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$.

$A_1 P$ and $A_2 P$ are symmetric if

$$\left\{ \begin{array}{l} (\frac{\lambda}{2} - a)v = -bu - \frac{\lambda}{2} v \\ (-\frac{\lambda}{2} - a)w = -\frac{\lambda}{2} u - bv \end{array} \right. \iff \left\{ \begin{array}{l} v = \frac{-b}{\lambda - a} u \\ w = \frac{\lambda(\lambda - a) - 2b^2}{(\lambda + 2a)(\lambda - a)} u \end{array} \right.$$

If we choose $u = \frac{(\lambda+2a)(\lambda-a)}{2}$, the eigenvalues of P are :

$$\frac{1}{2}\left(\lambda^2 - a^2 - b^2 \pm \sqrt{(a^2+b^2)^2 + \lambda(-2a^3 + 6ab^2) + \lambda^2(a^2+b^2)}\right).$$

We observe that p_1 and p_2 are equal to the eigenvalues of the diffusion matrix. Therefore, the matrix P is positive-definite if the stability conditions are verified.

Hyperbolicity condition $D2Q4$ (demonstration)

According to the equivalent equations, we have

$$\partial_t \boldsymbol{g} + A_1 \partial_{x_1} \boldsymbol{g} + A_2 \partial_{x_2} \boldsymbol{g} = 0$$

with $A_1 = \begin{pmatrix} -a & 0 & \frac{1}{2} \\ -b & 0 & 0 \\ \lambda^2 & 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -a & 0 \\ 0 & -b & -\frac{1}{2} \\ 0 & -\lambda^2 & 0 \end{pmatrix}$.

Demonstration : We note $P = \begin{pmatrix} u & v & w \\ v & x & y \\ w & y & z \end{pmatrix}$.

$A_1 P$ and $A_2 P$ are symmetric if

$$\begin{cases} bu = av - \frac{1}{2} y \\ \lambda^2 u = -aw + \frac{1}{2} z \\ \lambda^2 v = -bw \\ bv + \frac{1}{2} w = ax \\ \lambda^2 v = ay \\ \lambda^2 x = by + \frac{1}{2} z \end{cases}$$

Hyperbolicity condition $D2Q4$ (demonstration)

By solving this system and choosing $v = -ab$, we obtain

$$P = \begin{pmatrix} \frac{\lambda^2}{2} - a^2 & -ab & \lambda^2 a \\ -ab & \frac{\lambda^2}{2} - b^2 & -\lambda^2 b \\ \lambda^2 a & -\lambda^2 b & \lambda^4 \end{pmatrix}.$$

As P is symmetric, according to the Sylvester's criterion, P is positive-definite if and only if all of the leading principal minors are positive.

- $|P_1| = \frac{\lambda^2}{2} - a^2 > 0$ if the stability condition is verified.
- $|P_2| = \left(\frac{\lambda^2}{2} - a^2\right)\left(\frac{\lambda^2}{2} - b^2\right) - a^2 b^2 = \frac{\lambda^2}{2}\left(\frac{\lambda^2}{2} - a^2 - b^2\right) > 0.$
- $|P_3| = \lambda^4 \left(\frac{\lambda^2}{2} - 2a^2\right)\left(\frac{\lambda^2}{2} - 2b^2\right) > 0$ if $\frac{\lambda^2}{2} > 2\max\left(a^2, b^2\right)$ or if $\frac{\lambda^2}{2} > 2\min\left(a^2, b^2\right)$
  If the stability condition is verified, we cannot have $\frac{\lambda^2}{2} > 2\min\left(a^2, b^2\right)$.

Finally, the system is hyperbolic if $2\max\left(a^2, b^2\right) < \frac{\lambda^2}{2}$.